ENUMERATION OF SOME PARTICULAR QUINTUPLE PERSYMMETRIC MATRICES OVER $\mathbb{F}_2$ BY RANK

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Résumé. Dans cet article nous comptons le nombre de certaines quintuples matrices persymétriques de rang i sur $\mathbb{F}_2$.

Abstract. In this paper we count the number of some particular quintuple persymmetric rank i matrices over $\mathbb{F}_2$.

Contents

1. Introduction

In this paper we propose to compute in the most simple case the number of quintuple persymmetric matrices with entries in $\mathbb{F}_2$ of rank i

That is to compute the number $\Gamma_i^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k}$ of quintuple persymmetric matrices in $\mathbb{F}_2$ of rank i $(0 \leqslant i \leqslant \inf(10,k))$ of the below form.

$$
\left(\begin{array}{ccccccc}
\alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & \ldots & \alpha_k^{(1)} \\
\alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & \alpha_6^{(1)} & \ldots & \alpha_{k+1}^{(1)} \\
\hline
\alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \ldots & \alpha_k^{(2)} \\
\alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \alpha_6^{(2)} & \ldots & \alpha_{k+1}^{(2)} \\
\hline
\alpha_1^{(3)} & \alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)} & \ldots & \alpha_k^{(3)} \\
\alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)} & \alpha_6^{(3)} & \ldots & \alpha_{k+1}^{(3)} \\
\hline
\alpha_1^{(4)} & \alpha_2^{(4)} & \alpha_3^{(4)} & \alpha_4^{(4)} & \alpha_5^{(4)} & \ldots & \alpha_k^{(4)} \\
\alpha_2^{(4)} & \alpha_3^{(4)} & \alpha_4^{(4)} & \alpha_5^{(4)} & \alpha_6^{(4)} & \ldots & \alpha_{k+1}^{(4)} \\
\hline
\alpha_1^{(5)} & \alpha_2^{(5)} & \alpha_3^{(5)} & \alpha_4^{(5)} & \alpha_5^{(5)} & \ldots & \alpha_k^{(5)} \\
\alpha_2^{(5)} & \alpha_3^{(5)} & \alpha_4^{(5)} & \alpha_5^{(5)} & \alpha_6^{(5)} & \ldots & \alpha_{k+1}^{(5)}
\end{array}\right) \tag{1.1}
$$

We remark that this paper is based on the results in the author's paper [12]

2. Notation and Preliminaries

2.1. Some notations concerning the field of Laurent Series $\mathbb{F}_2((T^{-1}))$.

We denote by $\mathbb{F}_2\big((T^{-1})\big) = \mathbb{K}$ the completion of the field $\mathbb{F}_2(T)$, the field of rational fonctions over the finite field $\mathbb{F}_2$, for the infinity valuation $\mathfrak{v} = \mathfrak{v}_\infty$ defined by $\mathfrak{v}\big(\frac{A}{B}\big) = degB - degA$ for each pair (A,B) of non-zero polynomials. Then every element non-zero t in $\mathbb{F}_2\big(\big(\frac{1}{T}\big)\big)$ can be expanded in a unique way in a convergent Laurent series $t = \sum_{j=-\infty}^{-\mathfrak{v}(t)} t_j T^j \; where \; t_j \in \mathbb{F}_2$. We associate to the infinity valuation $\mathfrak{v} = \mathfrak{v}_\infty$ the absolute value $|\cdot|_\infty$ defined by

$$|t|_\infty = |t| = 2^{-\mathfrak{v}(t)}.$$

We denote E the Character of the additive locally compact group $\mathbb{F}_2\big(\big(\frac{1}{T}\big)\big)$ defined by

$$E\big(\sum_{j=-\infty}^{-\mathfrak{v}(t)} t_j T^j\big) = \begin{cases} 1 & \text{if } t_{-1} = 0, \\ -1 & \text{if } t_{-1} = 1. \end{cases}$$

We denote $\mathbb{P}$ the valuation ideal in $\mathbb{K}$, also denoted the unit interval of $\mathbb{K}$, i.e. the open ball of radius 1 about 0 or, alternatively, the set of all Laurent

series

$$\sum_{i\geq 1}\alpha_i T^{-i} \quad (\alpha_i \in \mathbb{F}_2)$$

and, for every rational integer j, we denote by $\mathbb{P}_j$ the ideal $\{t \in \mathbb{K}|\ \mathfrak{v}(t) > j\}$.
The sets $\mathbb{P}_j$ are compact subgroups of the additive locally compact group $\mathbb{K}$.
All $t \in \mathbb{F}_2\Big(\Big(\frac{1}{T}\Big)\Big)$ may be written in a unique way as $t = [t] + \{t\}$, $[t] \in \mathbb{F}_2[T],\ \{t\} \in \mathbb{P}(= \mathbb{P}_0)$.
We denote by dt the Haar measure on $\mathbb{K}$ chosen so that

$$\int_{\mathbb{P}} dt = 1.$$

$$Let \quad (t_1, t_2, \ldots, t_n) = \big(\sum_{j=-\infty}^{-\nu(t_1)}\alpha_j^{(1)}T^j, \sum_{j=-\infty}^{-\nu(t_2)}\alpha_j^{(2)}T^j, \ldots, \sum_{j=-\infty}^{-\nu(t_n)}\alpha_j^{(n)}T^j\big) \in \mathbb{K}^n.$$

We denote ψ the Character on $(\mathbb{K}^n, +)$ defined by

$$\begin{aligned}\psi\big(\sum_{j=-\infty}^{-\nu(t_1)}\alpha_j^{(1)}T^j, \sum_{j=-\infty}^{-\nu(t_2)}\alpha_j^{(2)}T^j, \ldots, \sum_{j=-\infty}^{-\nu(t_n)}\alpha_j^{(n)}T^j\big) &= E\big(\sum_{j=-\infty}^{-\nu(t_1)}\alpha_j^{(1)}T^j\big)\cdot E\big(\sum_{j=-\infty}^{-\nu(t_2)}\alpha_j^{(2)}T^j\big)\cdots E\big(\sum_{j=-\infty}^{-\nu(t_n)}\alpha_j^{(n)}T^j\big) \\ &= \begin{cases} 1 & \text{if } \alpha_{-1}^{(1)} + \alpha_{-1}^{(2)} + \ldots + \alpha_{-1}^{(n)} = 0 \\ -1 & \text{if } \alpha_{-1}^{(1)} + \alpha_{-1}^{(2)} + \ldots + \alpha_{-1}^{(n)} = 1 \end{cases}\end{aligned}$$

2.2. **Some results concerning n-times persymmetric matrices over $\mathbb{F}_2$.**

$$Set \quad (t_1, t_2, \ldots, t_n) = \big(\sum_{i\geq 1}\alpha_i^{(1)}T^{-i}, \sum_{i\geq 1}\alpha_i^{(2)}T^{-i}, \sum_{i\geq 1}\alpha_i^{(3)}T^{-i}, \ldots, \sum_{i\geq 1}\alpha_i^{(n)}T^{-i}\big) \in \mathbb{P}^n.$$

Denote by $D^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k}(t_1, t_2, \ldots, t_n)$
the following $2n \times k$ n-times persymmetric matrix over the finite field $\mathbb{F}_2$

$$
\left(\begin{array}{cccccccc}
\alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & \alpha_6^{(1)} & \ldots & \alpha_k^{(1)} \\
\alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & \alpha_6^{(1)} & \alpha_7^{(1)} & \ldots & \alpha_{k+1}^{(1)} \\
\hline
\alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \alpha_6^{(2)} & \ldots & \alpha_k^{(2)} \\
\alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \alpha_6^{(2)} & \alpha_7^{(2)} & \ldots & \alpha_{k+1}^{(2)} \\
\hline
\alpha_1^{(3)} & \alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)} & \alpha_6^{(3)} & \ldots & \alpha_k^{(3)} \\
\alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)} & \alpha_6^{(3)} & \alpha_7^{(3)} & \ldots & \alpha_{k+1}^{(3)} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
\alpha_1^{(n)} & \alpha_2^{(n)} & \alpha_3^{(n)} & \alpha_4^{(n)} & \alpha_5^{(n)} & \alpha_6^{(n)} & \ldots & \alpha_k^{(n)} \\
\alpha_2^{(n)} & \alpha_3^{(n)} & \alpha_4^{(n)} & \alpha_5^{(n)} & \alpha_6^{(n)} & \alpha_7^{(n)} & \ldots & \alpha_{k+1}^{(n)}
\end{array}\right) \tag{2.1}
$$

We denote by $\Gamma_i^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k}$ the number of rank i n-times persymmetric matrices over $\mathbb{F}_2$ of the above form :

Let $f(t_1,t_2,\ldots,t_n)$ be the exponential sum in $\mathbb{P}^n$ defined by
$(t_1,t_2,\ldots,t_n)\in\mathbb{P}^n \longrightarrow$
$\displaystyle\sum_{degY\leq k-1}\sum_{degU_1\leq 1}E(t_1YU_1)\sum_{degU_2\leq 1}E(t_2YU_2)\ldots\sum_{degU_n\leq 1}E(t_nYU_n).$

Then

$$
f_k(t_1,t_2,\ldots,t_n) = 2^{2n+k-rank\left[D^{\left[\begin{smallmatrix}2\\ \vdots\\ 2\end{smallmatrix}\right]\times k}(t_1,t_2,\ldots,t_n)\right]}
$$

Hence the number denoted by $R_{q,n}^{(k)}$ of solutions

$(Y_1,U_1^{(1)},U_2^{(1)},\ldots,U_n^{(1)},Y_2,U_1^{(2)},U_2^{(2)},\ldots,U_n^{(2)},\ldots Y_q,U_1^{(q)},U_2^{(q)},\ldots,U_n^{(q)})\in(\mathbb{F}_2[T])^{(n+1)q}$

of the polynomial equations

$$
\begin{cases}
Y_1U_1^{(1)}+Y_2U_1^{(2)}+\ldots+Y_qU_1^{(q)}=0\\
Y_1U_2^{(1)}+Y_2U_2^{(2)}+\ldots+Y_qU_2^{(q)}=0\\
\vdots\\
Y_1U_n^{(1)}+Y_2U_n^{(2)}+\ldots+Y_qU_n^{(q)}=0
\end{cases}
$$

$$\Leftrightarrow \begin{pmatrix} U_1^{(1)} & U_1^{(2)} & \dots & U_1^{(q)} \\ U_2^{(1)} & U_2^{(2)} & \dots & U_2^{(q)} \\ \vdots & \vdots & \vdots & \vdots \\ U_n^{(1)} & U_n^{(2)} & \dots & U_n^{(q)} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

satisfying the degree conditions

$$degY_i \leq k-1, \quad degU_j^{(i)} \leq 1, \quad for \quad 1 \leq j \leq n \quad 1 \leq i \leq q$$

is equal to the following integral over the unit interval in $\mathbb{K}^n$

$$\int_{\mathbb{P}^n} f_k^q(t_1, t_2, \dots, t_n) dt_1 dt_2 \dots dt_n.$$

Observing that $f(t_1, t_2, \dots, t_n)$ is constant on cosets of $\prod_{j=1}^n \mathbb{P}_{k+1}$ in $\mathbb{P}^n$ the above integral is equal to

$$2^{q(2n+k)-(k+1)n} \sum_{i=0}^{\inf(2n,k)} \Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times k} 2^{-iq} = R_{q,n}^{(k)} \tag{2.2}$$

Recall that $R_{q,n}^{(k)}$ is equal to the number of solutions of the polynomial system

$$\begin{pmatrix} U_1^{(1)} & U_1^{(2)} & \dots & U_1^{(q)} \\ U_2^{(1)} & U_2^{(2)} & \dots & U_2^{(q)} \\ \vdots & \vdots & \vdots & \vdots \\ U_n^{(1)} & U_n^{(2)} & \dots & U_n^{(q)} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.3}$$

satisfying the degree conditions

$$degY_i \leq k-1, \quad degU_j^{(i)} \leq 1, \quad for \quad 1 \leq j \leq n \quad 1 \leq i \leq q$$

From (2.2) we obtain for q = 1

$$2^{k-(k-1)n} \sum_{i=0}^{\inf(2n,k)} \Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times k} 2^{-i} = R_{1,n}^{(k)} = 2^{2n} + 2^k - 1 \tag{2.4}$$

We have obviously

$$\sum_{i=0}^{k} \Gamma_i^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right] \times k} = 2^{(k+1)n} \tag{2.5}$$

From the fact that the number of rank one persymmetric matrices over $\mathbb{F}_2$ is equal to three we obtain using combinatorial methods :

$$\Gamma_1^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} = (2^n-1)\cdot 3 \tag{2.6}$$

For more details see Cherly [11,12]

2.3. **The case n=5.**

Set $(t_1,t_2,t_3,t_4,t_5) = \left(\sum_{i\geq 1}\alpha_i^{(1)}T^{-i}, \sum_{i\geq 1}\alpha_i^{(2)}T^{-i}, \sum_{i\geq 1}\alpha_i^{(3)}T^{-i}, \sum_{i\geq 1}\alpha_i^{(4)}T^{-i}, \sum_{i\geq 1}\alpha_i^{(5)}T^{-i}\right) \in \mathbb{P}^5.$

Denote by $D^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k}(t_1,t_2,t_3,t_4,t_5)$
the following $10\times k$ quintuple persymmetric matrix over the finite field $\mathbb{F}_2$

$$\left(\begin{array}{ccccccc}
\alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & \ldots & \alpha_k^{(1)} \\
\alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & \alpha_6^{(1)} & \ldots & \alpha_{k+1}^{(1)} \\
\hline
\alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \ldots & \alpha_k^{(2)} \\
\alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \alpha_6^{(2)} & \ldots & \alpha_{k+1}^{(2)} \\
\hline
\alpha_1^{(3)} & \alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)} & \ldots & \alpha_k^{(3)} \\
\alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)} & \alpha_6^{(3)} & \ldots & \alpha_{k+1}^{(3)} \\
\hline
\alpha_1^{(4)} & \alpha_2^{(4)} & \alpha_3^{(4)} & \alpha_4^{(4)} & \alpha_5^{(4)} & \ldots & \alpha_k^{(4)} \\
\alpha_2^{(4)} & \alpha_3^{(4)} & \alpha_4^{(4)} & \alpha_5^{(4)} & \alpha_6^{(4)} & \ldots & \alpha_{k+1}^{(4)} \\
\hline
\alpha_1^{(5)} & \alpha_2^{(5)} & \alpha_3^{(5)} & \alpha_4^{(5)} & \alpha_5^{(5)} & \ldots & \alpha_k^{(4)} \\
\alpha_2^{(5)} & \alpha_3^{(5)} & \alpha_4^{(5)} & \alpha_5^{(5)} & \alpha_6^{(5)} & \ldots & \alpha_{k+1}^{(4)}
\end{array}\right)$$

We denote by $\Gamma_i^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k}$ the number of rank i quintuple persymmetric matrices over $\mathbb{F}_2$ of the above form :

Let $f(t_1,t_2,t_3,t_4,t_5)$ be the exponential sum in $\mathbb{P}^5$ defined by
$(t_1,t_2,t_3,t_4,t_5) \in \mathbb{P}^5 \longrightarrow$
$$\sum_{degY\leq k-1}\sum_{degU_1\leq 1}E(t_1YU_1)\sum_{degU_2\leq 1}E(t_2YU_2)\sum_{degU_3\leq 1}E(t_3YU_3)\sum_{degU_4\leq 1}E(t_4YU_4)\sum_{degU_5\leq 1}E(t_5YU_5).$$

Then

$$f_k(t_1,t_2,t_3,t_4,t_5) = 2^{10+k-rank\left[D^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k}(t_1,t_2,t_3,t_4,t_5)\right]}$$

Hence the number denoted by $R_{q,5}^{(k)}$ of solutions

$(Y_1, U_1^{(1)}, U_2^{(1)}, U_3^{(1)}, U_4^{(1)}, U_5^{(1)}, Y_2, U_1^{(2)}, U_2^{(2)}, U_3^{(2)}, U_4^{(2)}, U_5^{(2)}, \dots Y_q, U_1^{(q)}, U_2^{(q)}, U_3^{(q)}, U_4^{(q)}, U_5^{(q)}) \in (\mathbb{F}_2[T])^{6q}$

of the polynomial equations

$$\begin{cases} Y_1U_1^{(1)} + Y_2U_1^{(2)} + \ldots + Y_qU_1^{(q)} = 0 \\ Y_1U_2^{(1)} + Y_2U_2^{(2)} + \ldots + Y_qU_2^{(q)} = 0 \\ Y_1U_3^{(1)} + Y_3U_3^{(2)} + \ldots + Y_qU_3^{(q)} = 0 \\ Y_1U_4^{(1)} + Y_2U_4^{(2)} + \ldots + Y_qU_4^{(q)} = 0 \\ Y_1U_5^{(1)} + Y_2U_5^{(2)} + \ldots + Y_qU_5^{(q)} = 0 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} U_1^{(1)} & U_1^{(2)} & \ldots & U_1^{(q)} \\ U_2^{(1)} & U_2^{(2)} & \ldots & U_2^{(q)} \\ U_3^{(1)} & U_3^{(2)} & \ldots & U_3^{(q)} \\ U_4^{(1)} & U_4^{(2)} & \ldots & U_4^{(q)} \\ U_5^{(1)} & U_5^{(2)} & \ldots & U_5^{(q)} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

satisfying the degree conditions

$$degY_i \leq k-1, \quad degU_j^{(i)} \leq 1, \quad for \quad 1 \leq j \leq 5 \quad 1 \leq i \leq q$$

is equal to the following integral over the unit interval in $\mathbb{K}^5$

$$\int_{\mathbb{P}^5} f_k^q(t_1, t_2, t_3, t_4, t_5)dt_1dt_2dt_3dt_4dt_5$$

Observing that $f(t_1, t_2, t_3, t_4, t_5)$ is constant on cosets of $\prod_{j=1}^5 \mathbb{P}_{k+1}$ in $\mathbb{P}^5$ the above integral is equal to

$$2^{q(10+k)-5(k+1)} \sum_{i=0}^{\inf(10,k)} \Gamma_i^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right] \times k} 2^{-iq} = R_{q,5}^{(k)} \quad \text{where } k \geqslant 1 \tag{2.7}$$

We shall need the following results.

Result 1 :
We have whenever $k \geqslant 4$: See Cherly [12]

(2.8)

$$\Gamma_i^{\left[\begin{smallmatrix}2\\2\\2\\2\end{smallmatrix}\right]\times k} = \begin{cases} 1 & \text{if } i = 0, \\ 45 & \text{if } i = 1, \\ 30 \cdot 2^k + 1410 & \text{if } i = 2, \\ 1470 \cdot 2^k + 31920 & \text{if } i = 3, \\ 140 \cdot 2^{2k} + 42420 \cdot 2^k + 276640 & \text{if } i = 4, \\ 6300 \cdot 2^{2k} + 630000 \cdot 2^k - 11692800 & \text{if } i = 5, \\ 120 \cdot 2^{3k} + 123480 \cdot 2^{2k} - 6142080 \cdot 2^k + 66170880 & \text{if } i = 6. \\ 3720 \cdot 2^{3k} - 416640 \cdot 2^{2k} + 13332480 \cdot 2^k - 121896960 & \text{if } i = 7. \\ 16 \cdot 2^{4k} - 3840 \cdot 2^{3k} + 286720 \cdot 2^{2k} - 7864320 \cdot 2^k + 2^{26} & \text{if } i = 8. \end{cases}$$

where $\Gamma_i^{\left[\begin{smallmatrix}2\\2\\2\\2\end{smallmatrix}\right]\times k}$ denotes the number of quadruple persymmetric matrices in $\mathbb{F}_2$ of rank i $(0 \leqslant i \leqslant \inf(8, k))$ of the below form.

$$\left(\begin{array}{ccccccc} \alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & \ldots & \alpha_k^{(1)} \\ \alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & \alpha_6^{(1)} & \ldots & \alpha_{k+1}^{(1)} \\ \hline \alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \ldots & \alpha_k^{(2)} \\ \alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \alpha_6^{(2)} & \ldots & \alpha_{k+1}^{(2)} \\ \hline \alpha_1^{(3)} & \alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)} & \ldots & \alpha_k^{(3)} \\ \alpha_2^{(3)} & \alpha_3^{(3)} & \alpha_4^{(3)} & \alpha_5^{(3)} & \alpha_6^{(3)} & \ldots & \alpha_{k+1}^{(3)} \\ \hline \alpha_1^{(4)} & \alpha_2^{(4)} & \alpha_3^{(4)} & \alpha_4^{(4)} & \alpha_5^{(4)} & \ldots & \alpha_k^{(4)} \\ \alpha_2^{(4)} & \alpha_3^{(4)} & \alpha_4^{(4)} & \alpha_5^{(4)} & \alpha_6^{(4)} & \ldots & \alpha_{k+1}^{(4)} \end{array}\right)$$

Result 2

The $\Gamma_i^{\left[\begin{smallmatrix}2\\ \vdots\\2\end{smallmatrix}\right]\times k}$ where $0 \leq i \leq \inf(2n, k)$ (see subsection 2.2) are solutions to the below system. See Cherly[12]

(2.9)

$$\begin{cases}
\Gamma_{0}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} = 1 \quad \text{if} \quad k \geqslant 1 \\
\Gamma_{1}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} = (2^{n}-1)\cdot 3 \quad \text{if} \quad k \geqslant 2 \\
\Gamma_{2}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} = 7\cdot 2^{2n} + (2^{k+1}-25)\cdot 2^{n} - 2^{k+1} + 18 \quad \text{for} \quad k \geqslant 3 \\
\Gamma_{3}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} = 15\cdot 2^{3n} + (7\cdot 2^{k}-133)\cdot 2^{2n} + (294-21\cdot 2^{k})\cdot 2^{n} - 176 + 14\cdot 2^{k} \quad \text{for} \quad k \geqslant 4 \\
\Gamma_{4}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} = 31\cdot 2^{4n} + \dfrac{35\cdot 2^{k}-1210}{2}\cdot 2^{3n} + \dfrac{2^{2k+2}-783\cdot 2^{k}+19028}{6}\cdot 2^{2n} \\
+(-2^{2k+1}+269\cdot 2^{k}-5744)\cdot 2^{n} + \dfrac{2^{2k+2}-117\cdot 2^{k+2}+9440}{3} \quad \text{for} \quad k \geqslant 5 \\
\Gamma_{5}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} = 63\cdot 2^{5n} + (\dfrac{155}{4}\cdot 2^{k}-2573)\cdot 2^{4n} + (\dfrac{5}{2}\cdot 2^{2k} - \dfrac{2565}{4}\cdot 2^{k}+29150)\cdot 2^{3n} \\
+\dfrac{1}{2}\cdot(-35\cdot 2^{2k}+6265\cdot 2^{k}-247520)\cdot 2^{2n} + (35\cdot 2^{2k}-5490\cdot 2^{k}+203872)\cdot 2^{n} \\
-20\cdot 2^{2k}+2960\cdot 2^{k}-106752 \quad \text{for} \quad k \geqslant 6 \\
\displaystyle\sum_{i=0}^{\inf(2n,k)} \Gamma_{i}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} = 2^{(k+1)n} \\
\displaystyle\sum_{i=0}^{\inf(2n,k)} \Gamma_{i}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} 2^{-i} = 2^{n+k(n-1)} + 2^{(k-1)n} - 2^{(k-1)n-k} \\
\displaystyle\sum_{i=0}^{\inf(2n,k)} \Gamma_{i}^{\left[\begin{smallmatrix} 2 \\ \vdots \\ 2 \end{smallmatrix}\right]\times k} 2^{-2i} = 2^{n+k(n-2)} + 2^{-n+k(n-2)}\cdot[3\cdot 2^{k}-3] + 2^{-2n+k(n-2)}\cdot[6\cdot 2^{k-1}-6] \\
+2^{-3n+kn} - 6\cdot 2^{n(k-3)-k} + 8\cdot 2^{-3n+k(n-2)}
\end{cases}$$

Result 3
The number of rank 10 quintuple persymmetric matrices of the form (1.1) is equal to :

$2^{5}\prod_{j=1}^{5}(2^{k}-2^{10-j})$.See Cherly[10, section 2]

That is :

$$\Gamma_{10}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 2^5\prod_{j=1}^{5}(2^k-2^{10-j}) \tag{2.10}$$

2.4. **Computation of the number of quintuple persymmetric matrices of the form (1.1) of rank I.**

Theorem 2.1. *We have whenever $k \geqslant 5$:*

(2.11)
$$\begin{cases}
\Gamma_{0}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 1 \quad if \quad k \geqslant 1\\
\Gamma_{1}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 93 \quad if \quad k \geqslant 2\\
\Gamma_{2}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31\cdot 2^{k+1}+6386 \quad for \quad k \geqslant 3\\
\Gamma_{3}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 6510\cdot 2^{k}+364560 \quad for \quad k \geqslant 4\\
\Gamma_{4}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 620\cdot 2^{2k}+448260\cdot 2^{k}+15748000 \quad for \quad k \geqslant 5\\
\Gamma_{5}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 65100\cdot 2^{2k}+22654800\cdot 2^{k}+250817280 \quad for \quad k \geqslant 6\\
\Gamma_{6}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 1240\cdot [2^{3k}+3199\cdot 2^{2k}+2^7\cdot 2913\cdot 2^{k}-18883\cdot 2^{10}] \quad for \quad k \geqslant 7\\
\Gamma_{7}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 115320\cdot [2^{3k}+1148\cdot 2^{2k}-2^7\cdot 917\cdot 2^{k}+311\cdot 2^{13}] \quad for \quad k \geqslant 8\\
\Gamma_{8}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 496\cdot [2^{4k}+9525\cdot 2^{3k}-2169440\cdot 2^{2k}+68115\cdot 2^{11}\cdot 2^{k}-9749\cdot 2^{18}] \quad for \quad k \geqslant 9\\
\Gamma_{9}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31248\cdot [2^{4k}-480\cdot 2^{3k}+71680\cdot 2^{2k}-3932160\cdot 2^{k}+2^{26}] \quad for \quad k \geqslant 10\\
\Gamma_{10}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 2^5\cdot [2^{5k}-992\cdot 2^{4k}+317440\cdot 2^{3k}-40632320\cdot 2^{2k}+2080374784\cdot 2^{k}-2^{35}]\; for\; k \geqslant 10
\end{cases}$$

Proof. **Step 1**

From the expressions of $\Gamma_{i}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k}$ in (2.8) we postulate that
(2.12)

$$\Gamma_{i}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = \begin{cases} 1 & \text{if } i = 0, \\ a_1 & \text{if } i = 1, \\ a_2 \cdot 2^k + b_2 & \text{if } i = 2, \\ a_3 \cdot 2^k + b_3 & \text{if } i = 3, \\ a_4 \cdot 2^{2k} + b_4 \cdot 2^k + c_4 & \text{if } i = 4, \\ a_5 \cdot 2^{2k} + b_5 \cdot 2^k + c_5 & \text{if } i = 5, \\ a_6 \cdot 2^{3k} + b_6 \cdot 2^{2k} + c_6 \cdot 2^k + d_6 & \text{if } i = 6. \\ a_7 \cdot 2^{3k} + b_7 \cdot 2^{2k} + c_7 \cdot 2^k + d_7 & \text{if } i = 7. \\ a_8 \cdot 2^{4k} + b_8 \cdot 2^{3k} + c_8 \cdot 2^{2k} + d_8 \cdot 2^k + e_8 & \text{if } i = 8. \\ a_9 \cdot 2^{4k} + b_9 \cdot 2^{3k} + c_9 \cdot 2^{2k} + d_9 \cdot 2^k + e_9 & \text{if } i = 9. \\ a_{10} \cdot 2^{5k} + b_{10} \cdot 2^{4k} + c_{10} \cdot 2^{3k} + d_{10} \cdot 2^{2k} + e_{10} \cdot 2^k + f_{10} & \text{if } i = 10. \end{cases}$$

Step 2

Equally we postulate that :

$$\begin{cases} \Gamma_{6}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 0 \quad \text{for} \quad k = 5 \\ \Gamma_{7}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 0 \quad \text{for} \quad k \in \{5, 6\} \\ \Gamma_{8}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 0 \quad \text{for} \quad k \in \{5, 6, 7\} \\ \Gamma_{9}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 0 \quad \text{for} \quad k \in \{5, 6, 7, 8\} \\ \Gamma_{10}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 0 \quad \text{for} \quad k \in \{5, 6, 7, 8, 9\} \end{cases} \tag{2.13}$$

Step 3 Combining (2.9) with n=5 , (2.10) and (2.12) we obtain :

(2.14)

$$\begin{cases}
\Gamma_{0}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 1 \quad \text{for} \quad k \geqslant 1 \\
\Gamma_{1}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 93 \quad \text{for} \quad k \geqslant 2 \\
\Gamma_{2}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31\cdot 2^{k+1} + 6386 \quad \text{for} \quad k \geqslant 3 \\
\Gamma_{3}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 6510\cdot 2^{k} + 364560 \quad \text{for} \quad k \geqslant 4 \\
\Gamma_{4}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 620\cdot 2^{2k} + 448260\cdot 2^{k} + 15748000 \quad \text{for} \quad k \geqslant 5 \\
\Gamma_{5}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 65100\cdot 2^{2k} + 22654800\cdot 2^{k} + 250817280 \quad \text{for} \quad k \geqslant 6 \\
\Gamma_{6}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = a_6\cdot 2^{3k} + b_6\cdot 2^{2k} + c_6\cdot 2^{k} + d_6 \quad \text{for} \quad k \geqslant 7 \\
\Gamma_{7}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = a_7\cdot 2^{3k} + b_7\cdot 2^{2k} + c_7\cdot 2^{k} + d_7 \quad \text{for} \quad k \geqslant 8 \\
\Gamma_{8}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = a_8\cdot 2^{4k} + b_8\cdot 2^{3k} + c_8\cdot 2^{2k} + d_8\cdot 2^{k} + e_8 \quad \text{for} \quad k \geqslant 9 \\
\Gamma_{9}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = a_9\cdot 2^{4k} + b_9\cdot 2^{3k} + c_9\cdot 2^{2k} + d_9\cdot 2^{k} + e_9 \quad \text{for} \quad k \geqslant 10 \\
\Gamma_{10}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 2^{5}\cdot [2^{5k} - 992\cdot 2^{4k} + 317440\cdot 2^{3k} - 40632320\cdot 2^{2k} + 2080374784\cdot 2^{k} - 2^{35}] \text{ for } k \geqslant 10
\end{cases}$$

and the relations:

$$\begin{cases} \sum_{i=0}^{10} \Gamma_{i}^{\left[\begin{smallmatrix} 2\\ 2\\ 2\\ 2\\ 2 \end{smallmatrix}\right]\times k} = 2^{5k+5} \\ \sum_{i=0}^{10} \Gamma_{i}^{\left[\begin{smallmatrix} 2\\ 2\\ 2\\ 2\\ 2 \end{smallmatrix}\right]\times k} 2^{10-i} = 2^{5k+5} + 1023 \cdot 2^{4k+5} \\ \sum_{i=0}^{10} \Gamma_{i}^{\left[\begin{smallmatrix} 2\\ 2\\ 2\\ 2\\ 2 \end{smallmatrix}\right]\times k} 2^{20-2i} = 2^{5k+5} + 3162 \cdot 2^{4k+5} + 1045320 \cdot 2^{3k+5} \end{cases} \tag{2.15}$$

Step 4

Computation of a_8, a_9 in (2.14).
From (2.14) and(2.15) we get:

$$\begin{pmatrix} 1 & 1 \\ 4 & 2 \\ 16 & 4 \end{pmatrix} \begin{pmatrix} a_8 \\ a_9 \end{pmatrix} = \begin{pmatrix} 992 \cdot 2^5 \\ 992 \cdot 2^5 + 1023 \cdot 2^5 \\ 992 \cdot 2^5 + 3162 \cdot 2^5 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} a_8 \\ a_9 \end{pmatrix} = \begin{pmatrix} 496 \\ 31248 \end{pmatrix} \tag{2.16}$$

Step 5

Computation of $\Gamma_{9}^{\left[\begin{smallmatrix} 2\\ 2\\ 2\\ 2\\ 2 \end{smallmatrix}\right]\times k}$ in (2.14).
From (2.13) and (2.16) we obtain :

(2.17)
$$\Gamma_{9}^{\left[\begin{smallmatrix} 2\\ 2\\ 2\\ 2\\ 2 \end{smallmatrix}\right]\times k} = a_9(2^k-2^5)(2^k-2^6)(2^k-2^7)(2^k-2^8) = 31248\cdot[2^{4k}-480\cdot 2^{3k}+71680\cdot 2^{2k}-3932160\cdot 2^k+2^{26}]$$

To sum up we deduce from (2.17),(2.16) and (2.14)

(2.18)
$$\begin{cases}
\Gamma_{0}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 1 \quad \text{for} \quad k \geqslant 1 \\
\Gamma_{1}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 93 \quad \text{for} \quad k \geqslant 2 \\
\Gamma_{2}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31\cdot 2^{k+1} + 6386 \quad \text{for} \quad k \geqslant 3 \\
\Gamma_{3}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 6510\cdot 2^{k} + 364560 \quad \text{for} \quad k \geqslant 4 \\
\Gamma_{4}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 620\cdot 2^{2k} + 448260\cdot 2^{k} + 15748000 \quad \text{for} \quad k \geqslant 5 \\
\Gamma_{5}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 65100\cdot 2^{2k} + 22654800\cdot 2^{k} + 250817280 \quad \text{for} \quad k \geqslant 6 \\
\Gamma_{6}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = a_6\cdot 2^{3k} + b_6\cdot 2^{2k} + c_6\cdot 2^{k} + d_6 \quad \text{for} \quad k \geqslant 7 \\
\Gamma_{7}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = a_7\cdot 2^{3k} + b_7\cdot 2^{2k} + c_7\cdot 2^{k} + d_7 \quad \text{for} \quad k \geqslant 8 \\
\Gamma_{8}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 496\cdot 2^{4k} + b_8\cdot 2^{3k} + c_8\cdot 2^{2k} + d_8\cdot 2^{k} + e_8 \quad \text{for} \quad k \geqslant 9 \\
\Gamma_{9}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31248\cdot [2^{4k} - 480\cdot 2^{3k} + 71680\cdot 2^{2k} - 3932160\cdot 2^{k} + 2^{26}] \quad \text{for} \quad k \geqslant 10 \\
\Gamma_{10}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 2^5\cdot [2^{5k} - 992\cdot 2^{4k} + 317440\cdot 2^{3k} - 40632320\cdot 2^{2k} + 2080374784\cdot 2^{k} - 2^{35}] \text{ for } k \geqslant 10
\end{cases}$$

Step 6
Computation of a_6, a_7 and b_8 in (2.18).
From (2.18) and (2.15) we get:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2^4 & 2^3 & 2^2 \\ 2^8 & 2^6 & 2^4 \end{pmatrix} \begin{pmatrix} a_6 \\ a_7 \\ b_8 \end{pmatrix} = \begin{pmatrix} 31248\cdot 480 - 317440\cdot 2^5 \\ 2\cdot 31248\cdot 480 - 2^5\cdot 317440 \\ 2^2\cdot 31248\cdot 480 - 2^5\cdot 317440 + 1045320\cdot 2^5 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} a_6 \\ a_7 \\ b_8 \end{pmatrix} = \begin{pmatrix} 1240 \\ 115320 \\ 496 \cdot 9525 \end{pmatrix} \tag{2.19}$$

To sum up we deduce from (2.19),(2.18)

(2.20)

$$\begin{cases}
\Gamma_{0}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 1 \quad \text{for} \quad k \geqslant 1 \\
\Gamma_{1}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 93 \quad \text{for} \quad k \geqslant 2 \\
\Gamma_{2}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 31 \cdot 2^{k+1} + 6386 \quad \text{for} \quad k \geqslant 3 \\
\Gamma_{3}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 6510 \cdot 2^{k} + 364560 \quad \text{for} \quad k \geqslant 4 \\
\Gamma_{4}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 620 \cdot 2^{2k} + 448260 \cdot 2^{k} + 15748000 \quad \text{for} \quad k \geqslant 5 \\
\Gamma_{5}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 65100 \cdot 2^{2k} + 22654800 \cdot 2^{k} + 250817280 \quad \text{for} \quad k \geqslant 6 \\
\Gamma_{6}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 1240 \cdot 2^{3k} + b_6 \cdot 2^{2k} + c_6 \cdot 2^{k} + d_6 \quad \text{for} \quad k \geqslant 7 \\
\Gamma_{7}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 115320 \cdot 2^{3k} + b_7 \cdot 2^{2k} + c_7 \cdot 2^{k} + d_7 \quad \text{for} \quad k \geqslant 8 \\
\Gamma_{8}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 496 \cdot 2^{4k} + 496 \cdot 9525 \cdot 2^{3k} + c_8 \cdot 2^{2k} + d_8 \cdot 2^{k} + e_8 \quad \text{for} \quad k \geqslant 9 \\
\Gamma_{9}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 31248 \cdot [2^{4k} - 480 \cdot 2^{3k} + 71680 \cdot 2^{2k} - 3932160 \cdot 2^{k} + 2^{26}] \quad \text{for} \quad k \geqslant 10 \\
\Gamma_{10}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = 2^{5} \cdot [2^{5k} - 992 \cdot 2^{4k} + 317440 \cdot 2^{3k} - 40632320 \cdot 2^{2k} + 2080374784 \cdot 2^{k} - 2^{35}] \text{ for } k \geqslant 10
\end{cases}$$

Step 7

Computation of c_8, d_8 and e_8 in (2.20).
From (2.13) we get :

$$\begin{pmatrix} 2^{10} & 2^5 & 1 \\ 2^{12} & 2^6 & 1 \\ 2^{14} & 2^7 & 1 \end{pmatrix} \begin{pmatrix} c_8 \\ d_8 \\ e_8 \end{pmatrix} = \begin{pmatrix} -2^{15}\cdot 4740272 \\ -2^{18}\cdot 4756144 \\ -2^{21}\cdot 4787888 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} c_8 \\ d_8 \\ e_8 \end{pmatrix} = \begin{pmatrix} -2^{5}\cdot 33626320 \\ 2^{10}\cdot 67570080 \\ -2^{15}\cdot 38684032 \end{pmatrix} \tag{2.21}$$

Thus :

(2.22)

$$\begin{aligned}\Gamma_{8}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} &= 496\cdot 2^{4k} + 496\cdot 9525\cdot 2^{3k} - 2^{5}\cdot 33626320\cdot 2^{2k} + 2^{10}\cdot 67570080\cdot 2^{k} - 2^{15}\cdot 38684032 \\ &= 496\cdot [2^{4k} + 9525\cdot 2^{3k} - 2169440\cdot 2^{2k} + 68115\cdot 2^{11}\cdot 2^{k} - 9749\cdot 2^{18}]\end{aligned}$$

To sum up we deduce from (2.22),(2.20) :

(2.23)

$$\begin{cases}
\Gamma_0^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 1 \quad \text{for} \quad k \geqslant 1 \\
\Gamma_1^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 93 \quad \text{for} \quad k \geqslant 2 \\
\Gamma_2^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31 \cdot 2^{k+1} + 6386 \quad \text{for} \quad k \geqslant 3 \\
\Gamma_3^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 6510 \cdot 2^{k} + 364560 \quad \text{for} \quad k \geqslant 4 \\
\Gamma_4^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 620 \cdot 2^{2k} + 448260 \cdot 2^{k} + 15748000 \quad \text{for} \quad k \geqslant 5 \\
\Gamma_5^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 65100 \cdot 2^{2k} + 22654800 \cdot 2^{k} + 250817280 \quad \text{for} \quad k \geqslant 6 \\
\Gamma_6^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 1240 \cdot 2^{3k} + b_6 \cdot 2^{2k} + c_6 \cdot 2^{k} + d_6 \quad \text{for} \quad k \geqslant 7 \\
\Gamma_7^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 115320 \cdot 2^{3k} + b_7 \cdot 2^{2k} + c_7 \cdot 2^{k} + d_7 \quad \text{for} \quad k \geqslant 8 \\
\Gamma_8^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 496 \cdot [2^{4k} + 9525 \cdot 2^{3k} - 2169440 \cdot 2^{2k} + 68115 \cdot 2^{11} \cdot 2^{k} - 9749 \cdot 2^{18}] \quad \text{for} \quad k \geqslant 9 \\
\Gamma_9^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31248 \cdot [2^{4k} - 480 \cdot 2^{3k} + 71680 \cdot 2^{2k} - 3932160 \cdot 2^{k} + 2^{26}] \quad \text{for} \quad k \geqslant 10 \\
\Gamma_{10}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 2^5 \cdot [2^{5k} - 992 \cdot 2^{4k} + 317440 \cdot 2^{3k} - 40632320 \cdot 2^{2k} + 2080374784 \cdot 2^{k} - 2^{35}] \text{ for } k \geqslant 10
\end{cases}$$

Step 8

Computation of b_6, b_7 in (2.23).

From (2.15) and (2.23) we get :

$$\begin{pmatrix} 1 & 1 \\ 2^4 & 2^3 \\ 2^8 & 2^6 \end{pmatrix} \begin{pmatrix} b_6 \\ b_7 \end{pmatrix} = \begin{pmatrix} 136354120 \\ 1122567040 \\ 9488281600 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} b_6 \\ b_7 \end{pmatrix} = \begin{pmatrix} 1240 \cdot 3199 \\ 115320 \cdot 1148 \end{pmatrix} \tag{2.24}$$

To sum up we deduce from (2.23),(2.24) :

(2.25)

$$\begin{cases}
\Gamma_0^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 1 \quad \text{for} \quad k\geqslant 1\\
\Gamma_1^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 93 \quad \text{for} \quad k\geqslant 2\\
\Gamma_2^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31\cdot 2^{k+1}+6386 \quad \text{for} \quad k\geqslant 3\\
\Gamma_3^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 6510\cdot 2^{k}+364560 \quad \text{for} \quad k\geqslant 4\\
\Gamma_4^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 620\cdot 2^{2k}+448260\cdot 2^{k}+15748000 \quad \text{for} \quad k\geqslant 5\\
\Gamma_5^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 65100\cdot 2^{2k}+22654800\cdot 2^{k}+250817280 \quad \text{for} \quad k\geqslant 6\\
\Gamma_6^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 1240\cdot 2^{3k}+1240\cdot 3199\cdot 2^{2k}+c_6\cdot 2^{k}+d_6 \quad \text{for} \quad k\geqslant 7\\
\Gamma_7^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 115320\cdot 2^{3k}+115320\cdot 1148\cdot 2^{2k}+c_7\cdot 2^{k}+d_7 \quad \text{for} \quad k\geqslant 8\\
\Gamma_8^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 496\cdot [2^{4k}+9525\cdot 2^{3k}-2169440\cdot 2^{2k}+68115\cdot 2^{11}\cdot 2^{k}-9749\cdot 2^{18}] \quad \text{for} \quad k\geqslant 9\\
\Gamma_9^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31248\cdot [2^{4k}-480\cdot 2^{3k}+71680\cdot 2^{2k}-3932160\cdot 2^{k}+2^{26}] \quad \text{for} \quad k\geqslant 10\\
\Gamma_{10}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 2^5\cdot [2^{5k}-992\cdot 2^{4k}+317440\cdot 2^{3k}-40632320\cdot 2^{2k}+2080374784\cdot 2^{k}-2^{35}] \text{ for } k\geqslant 10
\end{cases}$$

Step 9

Computation of c_7, d_7 in (2.25).
From (2.13) we obtain :

$$\begin{pmatrix}32 & 1\\ 64 & 1\end{pmatrix}\begin{pmatrix}c_7\\ d_7\end{pmatrix} = \begin{pmatrix}-2^{15}\cdot 4252425\\ -2^{17}\cdot 4367745\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}c_7\\ d_7\end{pmatrix} = \begin{pmatrix}-2^{10}\cdot 13218555 = 115320\cdot(-2^7\cdot 917)\\ 2^{15}\cdot 8966130 = 115320\cdot(2^{12}\cdot 622)\end{pmatrix} \tag{2.26}$$

To sum up we deduce from (2.25),(2.26) :

(2.27)
$$\begin{cases}
\Gamma_{0}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 1 \quad \text{for} \quad k \geqslant 1 \\
\Gamma_{1}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 93 \quad \text{for} \quad k \geqslant 2 \\
\Gamma_{2}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31\cdot 2^{k+1} + 6386 \quad \text{for} \quad k \geqslant 3 \\
\Gamma_{3}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 6510\cdot 2^{k} + 364560 \quad \text{for} \quad k \geqslant 4 \\
\Gamma_{4}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 620\cdot 2^{2k} + 448260\cdot 2^{k} + 15748000 \quad \text{for} \quad k \geqslant 5 \\
\Gamma_{5}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 65100\cdot 2^{2k} + 22654800\cdot 2^{k} + 250817280 \quad \text{for} \quad k \geqslant 6 \\
\Gamma_{6}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 1240\cdot 2^{3k} + 1240\cdot 3199\cdot 2^{2k} + c_6\cdot 2^{k} + d_6 \quad \text{for} \quad k \geqslant 7 \\
\Gamma_{7}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 115320\cdot [2^{3k} + 1148\cdot 2^{2k} - 2^{7}\cdot 917\cdot 2^{k} + 2^{12}\cdot 622] \quad \text{for} \quad k \geqslant 8 \\
\Gamma_{8}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 496\cdot [2^{4k} + 9525\cdot 2^{3k} - 2169440\cdot 2^{2k} + 68115\cdot 2^{11}\cdot 2^{k} - 9749\cdot 2^{18}] \quad \text{for} \quad k \geqslant 9 \\
\Gamma_{9}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31248\cdot [2^{4k} - 480\cdot 2^{3k} + 71680\cdot 2^{2k} - 3932160\cdot 2^{k} + 2^{26}] \quad \text{for} \quad k \geqslant 10 \\
\Gamma_{10}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 2^{5}\cdot [2^{5k} - 992\cdot 2^{4k} + 317440\cdot 2^{3k} - 40632320\cdot 2^{2k} + 2080374784\cdot 2^{k} - 2^{35}] \text{ for } k \geqslant 10
\end{cases}$$

Step 10
Computation of c_6 in (2.27).
From (2.15) and (2.27) we get :

$$\begin{aligned}(2.28)\quad & 2^{k}\cdot\big(62 + 6510 + 448260 + 22654800 + c_6 + (-115320\cdot 2^{7}\cdot 917) + \\ & 496\cdot 68115\cdot 2^{11} + (-31248\cdot 3932160) + 2^{5}\cdot 2080374784\big) = 0 \\ & \Leftrightarrow \quad c_6 = 2^{10}\cdot 606515 = 1240\cdot 3913\cdot 2^{7}\end{aligned}$$

To sum up we deduce from (2.28),(2.27) :

(2.29)

$$\begin{cases}
\Gamma_{0}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 1 \quad \text{for} \quad k \geqslant 1 \\
\Gamma_{1}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 93 \quad \text{for} \quad k \geqslant 2 \\
\Gamma_{2}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31\cdot 2^{k+1} + 6386 \quad \text{for} \quad k \geqslant 3 \\
\Gamma_{3}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 6510\cdot 2^{k} + 364560 \quad \text{for} \quad k \geqslant 4 \\
\Gamma_{4}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 620\cdot 2^{2k} + 448260\cdot 2^{k} + 15748000 \quad \text{for} \quad k \geqslant 5 \\
\Gamma_{5}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 65100\cdot 2^{2k} + 22654800\cdot 2^{k} + 250817280 \quad \text{for} \quad k \geqslant 6 \\
\Gamma_{6}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 1240\cdot 2^{3k} + 1240\cdot 3199\cdot 2^{2k} + 1240\cdot 3913\cdot 2^{7}\cdot 2^{k} + d_6 \quad \text{for} \quad k \geqslant 7 \\
\Gamma_{7}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 115320\cdot [2^{3k} + 1148\cdot 2^{2k} - 2^{7}\cdot 917\cdot 2^{k} + 2^{12}\cdot 622] \quad \text{for} \quad k \geqslant 8 \\
\Gamma_{8}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 496\cdot [2^{4k} + 9525\cdot 2^{3k} - 2169440\cdot 2^{2k} + 68115\cdot 2^{11}\cdot 2^{k} - 9749\cdot 2^{18}] \quad \text{for} \quad k \geqslant 9 \\
\Gamma_{9}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 31248\cdot [2^{4k} - 480\cdot 2^{3k} + 71680\cdot 2^{2k} - 3932160\cdot 2^{k} + 2^{26}] \quad \text{for} \quad k \geqslant 10 \\
\Gamma_{10}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times k} = 2^{5}\cdot [2^{5k} - 992\cdot 2^{4k} + 317440\cdot 2^{3k} - 40632320\cdot 2^{2k} + 2080374784\cdot 2^{k} - 2^{35}] \text{ for } k \geqslant 10
\end{cases}$$

Step 11

Computation of d_6 in (2.29) :

From (2.13) we deduce :

$$\Gamma_{6}^{\left[\begin{smallmatrix}2\\2\\2\\2\\2\end{smallmatrix}\right]\times 5} = 0 \tag{2.30}$$

$$\Leftrightarrow \quad 1240\cdot 2^{15} + 1240\cdot 3199\cdot 2^{10} + 1240\cdot 3913\cdot 2^{7}\cdot 2^{5} + d_{6} = 0$$

$$\Leftrightarrow \quad d_{6} = -1240\cdot [2^{15} + 3199\cdot 2^{10} + 3913\cdot 2^{12}] = -1240\cdot 18883\cdot 2^{10}$$

and Theorem 2.1 is proved.

$\square$

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